Appendix: Mathiness in the Theory of Economic Growth by Paul Romer

This appendix exists as both a *Mathematica* notebook called Mathiness Appendix.nb and as two different pdf print-outs. The notebook is a combination of statements intended to communicate with people and statements that communicate with the computational engine in *Mathematica*.

One of the pdf printouts is called Mathiness Appendix.pdf. It prints only the plots and the typeset text that are intended as communication with people. The second pdf, called Mathiness Appendix Expanded.pdf prints those materials plus the supporting input to and detailed output from *Mathematica* that generates the plots and checks some of the algebra.

If you are reading one of these and would like any other, all three are both available for download from my website, paulromer.net.

Appendix A: Scale Effects

As a function of individual consumption q, the individual inverse demand function is

$$p = q^{-a},$$

so as a function of total quantity Q, the inverse demand function in a market with N identical individuals gives the demand price p_D as this function of Q and N:

$$p_D = \left(\frac{Q}{N}\right)^{-a}.$$

The market inverse supply function is

$$p_S = Q^b.$$

Implicitly, these expressions assume units for measuring both output Q and the numeraire good that set any multiplicative constant in these demand functions equal to 1.

The market clearing condition with a mark-up $m \geq 1$ that could arise from a tax or a monopoly markup is

$$p_D = m\, p_S,$$

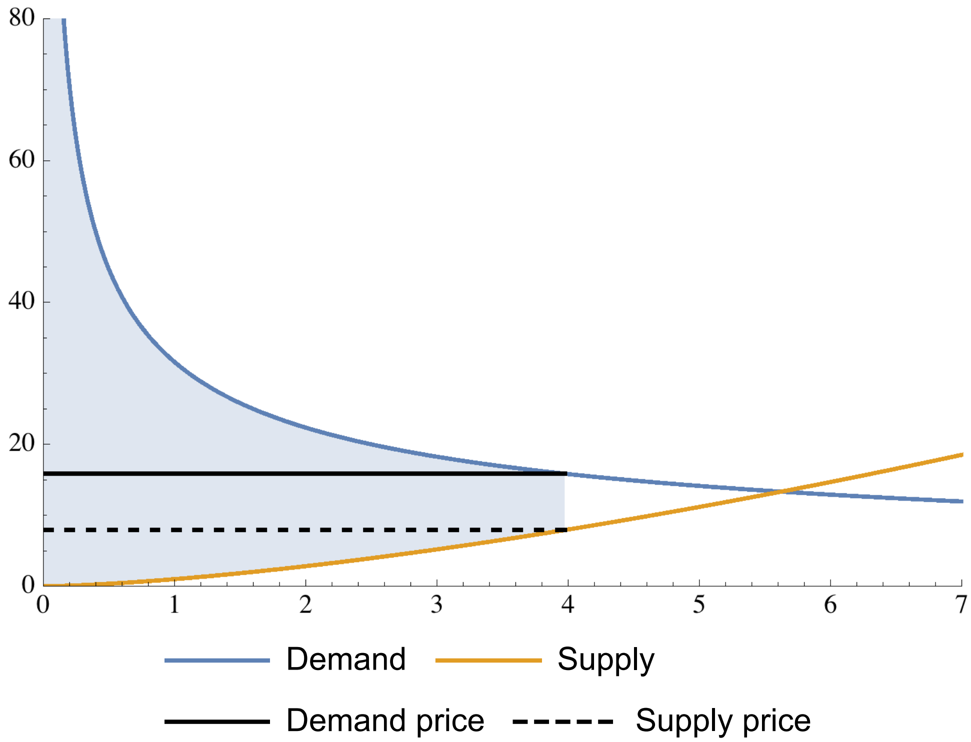
which implies

$$\left(\frac{Q}{N}\right)^{-a} = m\, Q^b.$$

This yields a solution for Q of the form

$$Q = m^{-\frac{1}{a+b}}\, N^{\frac{a}{a+b}}.$$

This figure illustrates an equilibrium with a mark-up $m = 2$, $a = 0.5$, $b = 1.5$, yields a value of Q roughly equal to 4:



The shaded area represents the surplus generated in this case. To calculate the value that this area represents, we start with the area under the individual demand curve from 0 to q, which takes the form

$$u(q) = \frac{1}{1-a} q^{1-a} .$$

As a function of the total quantity consumed, the sum across individuals of the area under the demand curves is

$$U(Q) = \frac{N}{1-a} \left(\frac{Q}{N}\right)^{1-a}.$$

Total cost, the area under the supply curve, is

$$C(Q) = \frac{1}{1+b} Q^{1+b} .$$

The surplus, equal to the shaded area in the figure, then can be calculated as

$$S = U(Q) - C(Q) = \frac{N}{1-a} \left(\frac{Q}{N}\right)^{1-a} - \frac{1}{1+b} Q^{1+b}.$$

The *Mathematica* calculations demonstrate that the implied expression for the surplus S as a function of the underlying parameters $a,\ b,\ m,$ and N is

$$S = \left(\frac{(a+m-1+b\,m)\ m^{-\frac{1+b}{a+b}}}{(1-a)\,(1+b)}\right) N^{\frac{a\,(1+b)}{a+b}}.$$

As noted in the main text, at $a = \frac{1}{2}$ and $b = 0$ this reduces to

$$S = \frac{2\,m-1}{m^2} N.$$

■ Calculations

■ Generating the Figure that Illustrates the Surplus

Appendix B: Growth in the Lucas-Moll Economies

To justify such statements as "the frequency at which innovations arrive, β, does not affect the growth rate, g" (Lucas and Moll 2014, p. 29), the authors rely on a limit argument about a limit does not exist.

If we let $g_\beta(t)$ denote the growth rate for a given value of β and date t, the rate g referenced in this quote is a limit,

$$g = \lim_{t \to \infty} g_\beta(t) = 2\,\%,$$

which indeed, does not depend on β. However, this number is not a good guide to the behavior specified in the model at any date T. The proof, available in the Expanded version of this appendix, shows that

$$\lim_{\beta \to 0} g_\beta(T) = 0.$$

In fact, it establishes a stronger result, that the growth rate converges uniformly to zero on any finite interval:

Proposition: *In the family of B economies, for any $T > 0$ and any $\epsilon > 0$, there exists a $\tilde{\beta} > 0$ such that for all $\beta \in \left(0, \tilde{\beta}\right)$ and all $t \in [0, T]$,*

$$0 \le g_\beta(t) < \epsilon.$$

So at every date T, the growth rate does depend on β and it converges to zero as β goes to zero. This is exactly what one should expect from the model itself. When β is set equal to 0, the model implies that the growth rate is also equal to zero.

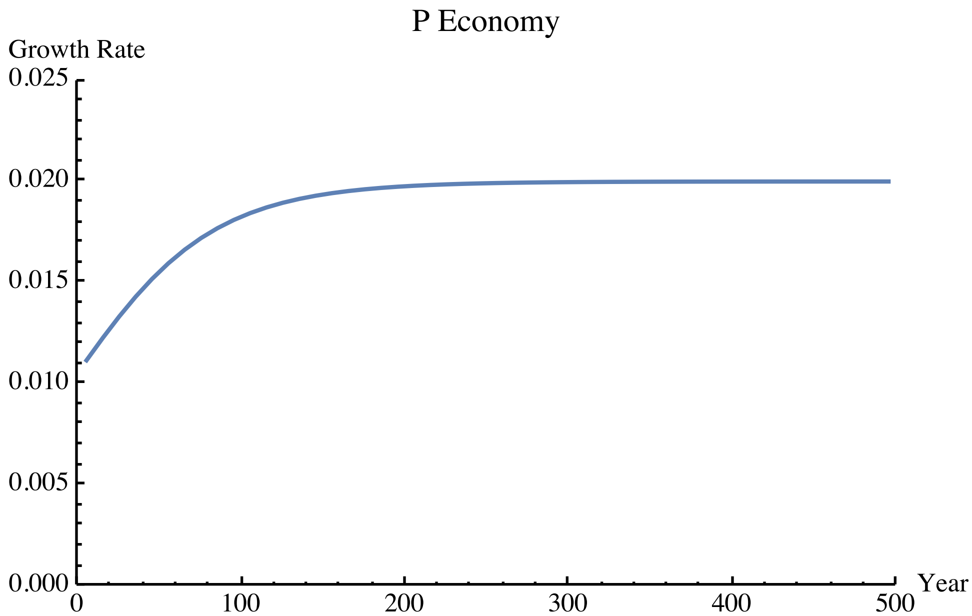
The difference between these two types of limit implies that the order one uses to calculate the double limit matters:

$$\lim_{T \to \infty} \lim_{\beta \to 0} g_\beta(T) \neq \lim_{\beta \to 0} \lim_{T \to \infty} g_\beta(T).$$

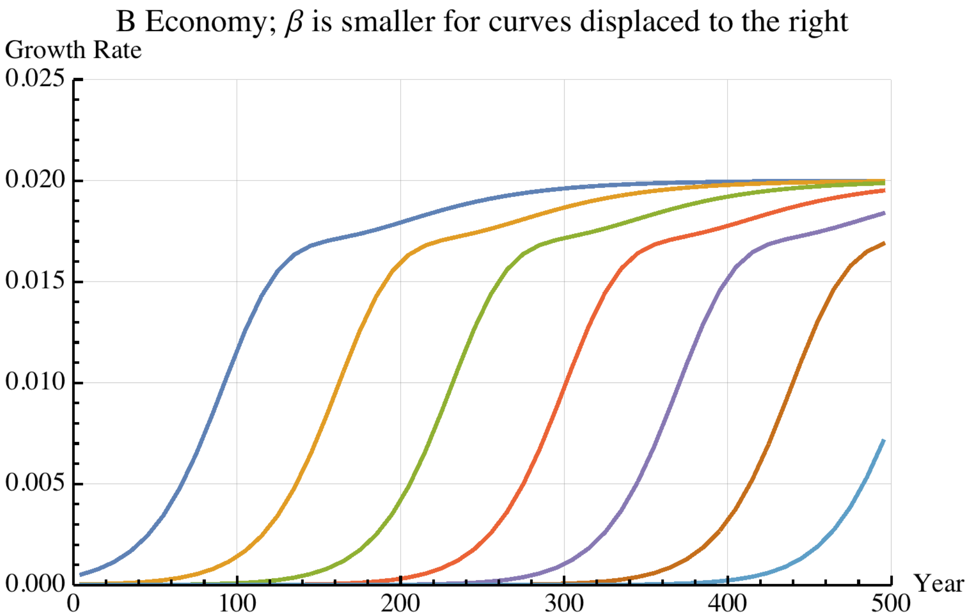
This dependence on the order tells us that if we recognize $g(\beta, T)$ as a function from $\mathbb{R}^2 \to \mathbb{R}$, the limit of g as (β, T) goes to $(0, \infty)$ does not exist.

Lucas and Moll use limit arguments about this limit that does not exist to claim that two types of economy, labeled here the P and B economies, are so similar as to be observationally equivalent. Simple numerical calculations show the growth rate in these two types of economy differ in ways that are plainly observable.

For example, this figure shows a plot over time of the growth rate in a version of the P economy.



Contrast this with the next figure, which shows the plot over time of the growth rate in a collection of B economies with decreasing arrival rates β, where curves displaced to the right have a lower value of β. It shows how the growth rate at any fixed date such as $T = 300$ falls from a value close to 2% to a value close to zero as β decreases.



These plots show that there at every date T is no value for the growth rate g that the P economy shares with the collection of all the B economies. The only value that they have in common is the limit of the growth rate for fixed β as T goes to infinity. As noted in the text of the accompanying article, this kind of limit is not an observable. No feasible set of observations on the growth rate, which must of course be taken at finite times, can ever falsify an assertion about a limit as time goes to infinity.

- **Plotting Growth in the P Economy**
- **Plotting Growth in the B Economies**

Appendix C: Calculating the Ratio of Wealth to Income

Let Y_γ denote income measured in gross terms and let Y_ν denote income measured in net terms. (The mnemonic is "gamma for gross" and "nu for net.") Symmetrically, let s_γ and s_ν denote the gross and net measures of saving. Let δ be the depreciation rate. Let g be the rate of growth of gross output. Let K be the stock of capital.

By definition,

$$Y_\nu = Y_\gamma - \delta K,$$

$$\dot{K} = s_\gamma Y_\gamma - \delta K = s_\nu Y_\nu.$$

For $\frac{K}{Y_\gamma}$ to be constant in a steady state, $\frac{\dot{K}}{K}$ must be equal to g, which implies

$$g = \frac{\dot{K}}{K} = s_\gamma \frac{Y_\gamma}{K} - \delta = s_\nu \frac{Y_\nu}{K}.$$

This yields the two expressions for the steady-state ratio of capital to output,

$$\frac{K}{Y_\gamma} = \frac{s_\gamma}{g + \delta}, \tag{1}$$

$$\frac{K}{Y_\nu} = \frac{s_\nu}{g}. \tag{2}$$

From the definitions, the ratio of Y_γ to Y_ν is equal to

$$\frac{Y_\gamma}{Y_\nu} = \frac{Y_\nu + \delta K}{Y_\nu} = 1 + \delta \frac{K}{Y_\nu} = 1 + \delta \frac{s_\nu}{g} \tag{3}$$

By taking the ratio of equation (2) to equation (1), we have a second expression for the ratio of gross to net income

$$\frac{Y_\gamma}{Y_\nu} = \frac{s_\nu}{s_\gamma} \frac{g + \delta}{g}. \tag{4}$$

Combining this with equation (3) yields

$$1 + \delta \frac{s_\nu}{g} = \frac{s_\nu}{s_\gamma} \frac{g + \delta}{g}.$$

Solving for s_γ yields

$$s_\gamma = s_\nu \frac{g + \delta}{g} \frac{1}{1 + \delta \frac{s_\nu}{g}} = s_\nu \frac{g + \delta}{g} \frac{g}{g + \delta\, s_\nu} = s_\nu \frac{g + \delta}{g + \delta\, s_\nu}. \tag{5}$$

Let g be an initial growth rate and g^* be a lower growth rate. In the example considered by Piketty and Zucman (2014), $g^* = \frac{1}{2} g$. Following their approach, assume that s_ν stays constant when g changes. Let other variables with an asterisk denote the values implied by the new saving rate. For example, let $s_\gamma{}^*$ denote the new gross saving rate implied by g^*.

Consider the ratio

$$\frac{g}{g^*} = \frac{\left(\frac{K}{Y_\nu}\right)^*}{\frac{K}{Y_\nu}}.$$

To decompose this into the three terms described in the text, rewrite the numerator and denominator in terms of ratios that involve Y_γ

$$\frac{g}{g^*} = \frac{\left(\frac{K}{Y_\gamma}\right)^*}{\left(\frac{K}{Y_\gamma}\right)} \frac{\left(\frac{Y_\gamma}{Y_\nu}\right)^*}{\frac{Y_\gamma}{Y_\nu}}.$$

Then use the expression for $\frac{K}{Y_\gamma}$ from equation (1) to rewrite the first term,

$$\frac{g}{g^*} = \frac{\frac{s_\gamma{}^*}{g^*+\delta}}{\frac{s_\gamma}{g+\delta}} \frac{\left(\frac{Y_\gamma}{Y_\nu}\right)^*}{\frac{Y_\gamma}{Y_\nu}}$$

and multiply and divide by s_γ to get

$$\frac{g}{g^*} = \left(\frac{\frac{s_\gamma}{g^*+\delta}}{\frac{s_\gamma}{g+\delta}}\right)\left(\frac{s_\gamma{}^*}{s_\gamma}\right)\left(\frac{\left(\frac{Y_\gamma}{Y_\nu}\right)^*}{\frac{Y_\gamma}{Y_\nu}}\right) = T_1\, T_2\, T_3.$$

The first of these three terms,

$$T_1 = \frac{\frac{s_\gamma}{g^*+\delta}}{\frac{s_\gamma}{g+\delta}},$$

is the change in the ratio of capital to gross income that we would observe if the gross saving rate remained constant at its initial value s_γ when the growth rate falls from g to g^*. The second term,

$$T_2 = \frac{{s_\gamma}^*}{s_\gamma} = \frac{\frac{g^*+\delta}{g^*+\delta s_\nu}}{\frac{g+\delta}{g+\delta s_\nu}}$$

captures the change in the gross saving rate that is implied by the change in g when s_ν remains constant. The final term,

$$T_3 = \frac{\left(\frac{Y_\gamma}{Y_\nu}\right)^*}{\frac{Y_\gamma}{Y_\nu}} = \frac{1+\delta\,\frac{s_\nu}{g^*}}{1+\delta\,\frac{s_\nu}{g}},$$

is the change in the ratio of gross to net income implied by the change in the growth rate.

The parameter values in the paper, $\delta = 3\ \%$, $s_\nu = 10\ \%$, $g = 3\ \%$, and $g^* = 1.5\ \%$ imply values

$$T_1 = 1.333,$$
$$T_2 = 1.375, \text{ and}$$
$$T_3 = 1.091.$$

In this example, because g falls to half its previous value, the ratio of K to Y_ν doubles. The calculations verify that with these values, the product $T_1\ T_2\ T_3$ is equal to 2.

■ Calculations for steady state relationships

■ Dynamics

Because it is an easy exercise, it may also be of some interest to explore the time path for these variables. Consider a case of an economy that starts in a steady state with $g = 3\ \%$ and $s_\nu = 10\ \%$ so the implied value for s_γ is approximately 18 %. We can calculate the capital to gross output ratio as

$$\frac{K}{Y_\gamma} = \frac{s_\gamma}{g + \delta}.$$

Let $k = \frac{K}{Y_\gamma}$ denote this ratio and consider the expression for its percentage rate of change

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{Y}_\gamma}{Y_\gamma} = s_\gamma \frac{Y_\gamma}{K} - \delta - g,$$

which implies

$$\dot{k} = s_\gamma - (\delta + g)\, k, \tag{6}$$

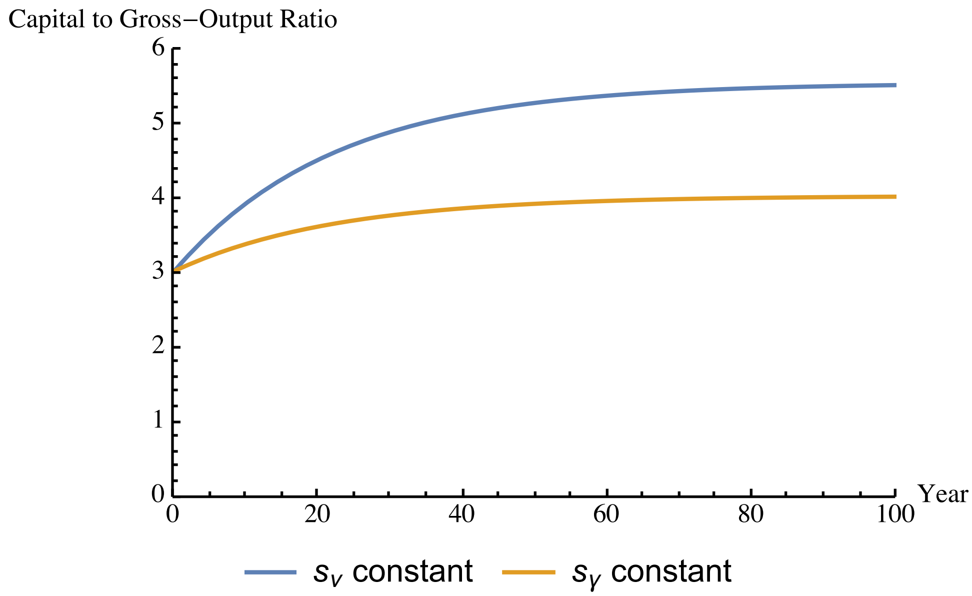
where s_γ is related to s_ν by equation (9) from above,

$$s_\gamma = s_\nu \frac{g + \delta}{g + \delta s_\nu}. \tag{7}$$

Suppose that at time zero, the economy is at its steady state at a growth rate $g = 3$ % per year. At time zero, the growth rate falls to from 3 % to 1.5 %.

In an approach that maintains that the saving rate is fixed when the growth rate changes, there are now two possibilities -- either the gross saving rate remains constant or the net saving rate remains constant. To capture the first case, we can use a version of equation (6) uses $g^* = 1.5$ % but leaves s_γ unchanged at roughly 18 %. In the second case, in equation (6) we insert both the new value $s_\gamma{}^* = 25$ % implied by equation (7) and the new growth rate g^*.

This plot compares the evolution of the capital to gross output ratio under these two scenarios.



■ Calculations for the dynamic paths